't Hooft GR
exercise 12.1
Pg 53

$$\varphi - \varphi_0 = \int_{u_0}^{u} du \left(\frac{E^2-1}{J^2} + \frac{2M}{J^2} u - u^2 \right)^{-1/2}$$

can be solved via

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| & a>0 \\ \frac{1}{\sqrt{-a}} \sin^{-1} \frac{-2ax-b}{\sqrt{b^2-4ac}} & a<0 \end{cases}$$

$$\Rightarrow \int_{u_0}^{u} du \left(\cdots \right)^{-1/2} = \sin^{-1} \frac{2u - \frac{2M}{J^2}}{\sqrt{\left(\frac{2M}{J^2}\right)^2 + 4\left(\frac{E^2-1}{J^2}\right)}}$$

$$= \boxed{\sin^{-1} \frac{\left(2u - \frac{2M}{J^2}\right) J}{\sqrt{\frac{2M}{J^2} - 4(E^2-1)}}}$$

In the Newtonian Limit, r is large, so $\frac{1}{r}$ small,

$u = \frac{1}{r}$ is small, u^3 is $(\text{small})^3$.

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5.30.2024

't Hooft GR 12.1 integral part

We wish to evaluate

$$\varphi - \varphi_0 = \int_{x_0}^{x} \frac{1}{\sqrt{A + Bx - x^2}}\, dx$$

where $A = \frac{E^2 - 1}{J^2}$, $B = \frac{2M}{J^2}$

Use $\frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$

Let $u = ax - b$, $u^2 = a^2x^2 - 2abx + b^2$

$$1 - u^2 = (1-b^2) + 2abx - a^2x^2$$

$$\frac{d}{dx} \arccos[u] = \frac{(-1)(a)}{\sqrt{(1-b^2) + (2ab)x - a^2x^2}}$$

$$= \frac{-1}{\sqrt{\frac{1-b^2}{a^2} + \frac{2b}{a}x - x^2}}$$

Plug in $A = \frac{E^2-1}{J^2} = \frac{1-b^2}{a^2}$, $B = \frac{2M}{J^2} = \frac{2b}{a}$

$\Rightarrow b = \frac{M}{J^2} a$, $\frac{1}{a^2} = \frac{E^2-1}{J^2} + \frac{M^2}{J^4}$

$\frac{1-b^2}{a^2} = \frac{E^2-1}{J^2}$,

$$\frac{1 - \frac{M^2}{J^4}a^2}{a^2} = \frac{E^2-1}{J^2},$$

$$\boxed{a = \frac{J^2}{\sqrt{J^2[E^2-1] + M^2}}}$$

$$\boxed{b = \frac{M}{\cancel{J^2}} \frac{\cancel{J^2}}{\sqrt{J^2[E^2-1] + M^2}}}$$

$$\Rightarrow \varphi - \varphi_0 = \int_{x_0}^{x} -\frac{d}{dx}\left[\arccos[ax - b]\right] dx$$

$$= -\arccos[ax - b]\Big|_{x_0}^{x}$$

where $a = \dfrac{J^2}{\sqrt{J^2[E^2 - 1] + M^2}}$

$b = \dfrac{M^2}{\sqrt{J^2[E^2 - 1] + M^2}}$

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7-4-2024